Solution of the non-linear equations of cellular convection and heat transport

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By expanding the dependent variables in series of orthogonal functions on the one hand, and expanding the coefficients of these functions in power series of a parameter η on the other hand, a solution has been obtained for the system of non-linear equations of cellular convection. The expansion parameter η is chosen in such a way as to make it remain less than 1 for all finite values of the Rayleigh number. The solution so obtained is found to be valid for a large range of the imposed temperature difference, and converges rapidly. This solution provides a quantitative theory for the convective heat transport as a function of the temperature difference in the range of laminar flow.

The solution also reveals that when the actual Rayleigh number is greater than twice the critical Rayleigh number, a layer of isothermal (adiabatic lapse-rate in a gas medium) mean temperature develops in the middle of the fluid layer. The thickness of this layer increases as the actual Rayleigh number increases, and at the same time the temperature gradient increases in the boundary layer so that an increase in the heat transport is accomplished.

The solution reveals further that the large temperature gradients are concentrated in the region where the cold descending current approaches the lower boundary and where the warm ascending current approaches the upper boundary. It is also shown that these ascending and descending currents spread out in mushroom-like patterns, a feature characteristic of the convection of isolated hot bubbles, but one which never has been considered as the form for finite cellular convection. Recent optical observations indicate that this is the most common form of the temperature field.

The heat transport given by this solution fits a power law of exponent 1·24, which is very close to the observed power law of exponent 1·25 for laminar flow.

1. Introduction

When a layer of fluid is heated uniformly from below and cooled from above, cellular convection starts to set in as the Rayleigh number R reaches its critical value R_0. This starting convection has a definite form and a definite scale, which are given by the solutions of the linearized equations in accordance with the boundary conditions (Rayleigh 1916; Pellew & Southwell 1940).

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When R is raised above R_0, the motion increases its intensity but remains laminar and steady for a large range of values of R, followed by unsteady, turbulent convection at a much higher temperature difference.

Formal solutions of the non-linear equations for thermal convection have been obtained in the works of Malkus & Veronis (1958) and Kuo & Platzman (1960), under the tacit assumptions that the motion is steady and that all the higher modes of convection are created by the non-linear interactions. However, the solutions obtained so far are valid only for small values of $(R-R_0)$, and therefore cannot be used for larger temperature differences.

It should be mentioned that by assuming a steady state we are considering the final equilibrium state between the temperature field and the motion field. If this equilibrium state is stable, it must be reached asymptotically. On the other hand, if it is unstable, it will be replaced by another state which is stable but is more likely to be unsteady.

In the work presented here a different solution of the steady non-linear equations is obtained. This new solution converges more rapidly and is valid for a much larger range of the imposed temperature difference. Because of these features, this solution provides a quantitative theory for the convective heat transport as a function of the temperature difference for a large range of the laminar flows. It also sheds some light on the problem of transition to turbulent convection, which occurs at a much higher temperature difference.

The method of solution adopted here is that of double expansions. In this method, the dependent variables are first expressed as infinite series of a set of orthogonal space functions, and second, the amplitudes of these functions are expressed as infinite series of an expansion parameter η which is a function of R.

For the case of two free boundaries the space functions of all modes are simple sine and cosine functions, and therefore the dependent variables can be represented directly by double Fourier trigonometric series. The non-linear equations are then reduced to spectral equations and the solutions can be obtained more readily by induction, as has been already done by Kuo & Platzman (1960). In the present paper, the same method will be used in obtaining the new solution for two free boundaries, and it will be developed to higher approximations.

For the cases of two rigid boundaries and of one rigid and one free boundary, these orthogonal functions must be obtained in a consecutive manner, as solutions of a set of linear but inhomogeneous equations, with the inhomogeneous terms containing functions of the lower modes. Therefore the method of solution for these cases is similar to that employed by Malkus & Veronis (1958), but differs from theirs in detail.

In this paper, we shall choose the expansion parameter η in such a way as to make it remain less than 1 for all finite values of R. The solution so obtained will be valid for a much larger range of temperature difference and will converge more rapidly.

Because of the difference in the methods of solution employed, only the solution for the case of two free boundaries will be presented in this paper, while those for the case of two rigid boundaries and of one rigid and one free boundary will be presented in another paper.

2. The governing equations

Consider a horizontal layer of fluid of depth h, confined between two parallel planes at $z = 0$ and $z = h$, the upper and lower planes being maintained at temperatures T_a and T_0 respectively. In the undisturbed state the temperature T is constant over a horizontal plane and decreases linearly with z and is given by

$$T = T_0 + \beta z, \tag{2.1}$$

where $\beta = (T_a - T_0)/h$ is the constant initial temperature gradient. The relation between the temperature T and the density ρ is given by

$$\rho = \rho_0\{1 - \alpha(T - T_0)\}, \tag{2.2}$$

where ρ_0 is the density corresponding to the temperature T_0 and α is the thermal expansion coefficient.

To investigate the convection problem, it is only necessary to take into consideration the variation of density when it is multiplied by the gravity g, whereby it introduces a buoyancy force $g\alpha T'$, where T' is the departure of the temperature from its horizontal average. At all other places, such as in the inertia terms of the equations of motion and in the continuity equation, the effect of the variations of ρ is quite small and can be neglected. Therefore we shall replace ρ by its mean value ρ_m after introducing the buoyancy force in the vertical equation of motion. This is the principal feature of the so-called Boussinesq approximation.

On making use of the Boussinesq approximation, the equations of motion, the thermal energy equation and the continuity equation may be written as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho_0}\nabla p + g\alpha T'\mathbf{k} + \nu\nabla^2\mathbf{v}, \tag{2.3}$$

$$\frac{dT}{dt} = \kappa\nabla^2 T, \tag{2.4}$$

$$\nabla . \mathbf{v} = 0, \tag{2.5}$$

where $\mathbf{v}$ is the vector velocity, $\mathbf{k}$ is the unit vector along the vertical, p is the departure of pressure from the hydrostatic pressure $\bar{p}(z)$, T' is the departure of temperature from the horizontal average $\bar{T}(z)$, ν is the kinematic viscosity coefficient, κ is the thermometric conductivity and α is the thermal expansion coefficient.

As in the previous study, we shall restrict our consideration in this paper to the simplest model of cellular convection: the infinite roll in a steady state, such that $\partial/\partial y = 0$ and $\partial/\partial t = 0$. It is then more convenient to introduce a stream function ψ to represent the field of motion, given by

$$u = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial x}. \tag{2.6}$$

The fundamental equations (2.3) to (2.5) can be reduced to non-dimensional forms by the following choice of the units of length, time, and temperature, respectively

$$h, \quad h^2/\kappa, \quad \kappa\nu/g\alpha h^3. \tag{2.7}$$

Henceforth the symbols u, ψ, T, p, etc., will represent the physical values they have had heretofore divided by the relevant dimensional quantities from (2.7).

Eliminating p from the equation of motion (2.3) by applying the curl-operator and expressing in terms of the non-dimensional stream function ψ we obtain the following steady-state vorticity equation

$$\nabla^4\psi + \frac{\partial\theta}{\partial x} = \frac{1}{\sigma}B, \quad B \equiv \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,z)}, \quad \sigma = \frac{\nu}{\kappa}, \tag{2.8}$$

where θ is defined below.

Expressing the temperature as the sum of the undisturbed mean temperature and the departure (θ) from this mean, then the total non-dimensional temperature T is given by

$$T = T_0 - Rz + \theta, \quad R \equiv \frac{g\alpha(\Delta T')}{\kappa\nu}h^3, \tag{2.9}$$

where $\Delta T'$ is the dimensional temperature difference between the bottom and the top, and therefore R is the Rayleigh number.

The non-dimensional steady-state thermal energy equation is given by

$$\nabla^2\theta + R\frac{\partial\psi}{\partial x} = H, \quad H \equiv \frac{\partial(\psi,\theta)}{\partial(x,z)}. \tag{2.10}$$

The equations (2.8) and (2.10) form a closed system for the two dependent variables ψ and θ.

The boundary conditions with respect to which these equations must be solved depend on whether the bounding surface is free or rigid. On a free surface, the vertical velocity and the tangential stress must vanish. Since the temperature of the boundary is being kept constant, we shall then have for a free surface,

$$\psi = \nabla^2\psi = \theta = 0. \tag{2.11}$$

On the other hand, if the surface is rigid, then both the vertical and the tangential velocities must vanish, and therefore we must have

$$\psi = \frac{\partial\psi}{\partial z} = \theta = 0. \tag{2.12}$$

An important consequence of these conditions is that both the vorticity advection Jacobian B and the heat advection Jacobian H vanish at the lower and upper boundaries.

3. Spectral equations

For the case of two free boundaries, all the space functions of the various modes $\psi(l,n)$ and $\theta(l,n)$ are sine and cosine functions,

$$\psi(l,n) \sim \sin(lk\pi x)\sin(n\pi z), \quad \theta(l,n) \sim \cos(lk\pi x)\sin(n\pi z), \tag{3.1}$$

where l and n are integers and k is the horizontal wave-number of the first mode ($l = n = 1$), which is geometrically unrestricted when the fluid extends to infinity in the horizontal direction. However, since convection is assumed to set in when R reaches its critical value R_0, which is a function of k, we must choose k so as to make R_0 a minimum. Solutions corresponding to other k-values represent unstable situations and therefore cannot persist.

Since all the functions (3.1) satisfy the boundary conditions (2.11) we may represent ψ and θ by the following double Fourier expansions

$$\left.\begin{aligned}\psi &= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\hat{\psi}_{l,n}\sin(lk\pi x)\sin(n\pi z),\\ \theta &= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\hat{\theta}_{l,n}\cos(lk\pi x)\sin(n\pi z),\end{aligned}\right\} \tag{3.2}$$

where $\hat{\psi}_{l,n}$ and $\hat{\theta}_{l,n}$ are functions of R.

In order to transform the basic non-linear differential equations (2.8) and (2.10) into spectral forms, it is more convenient to express ψ and θ in the following complex forms in place of (3.2)

$$\left.\begin{aligned}\psi &= -\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\psi_{l,n}\exp(lkx+nz)\pi i,\\ \theta &= -\pi^3 i\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\theta_{l,n}\exp(lkx+nz)\pi i.\end{aligned}\right\} \tag{3.3}$$

We note that (3.2) is equivalent to (3.3) provided

$$\left.\begin{aligned}\psi_{l,n} &= -\psi_{-l,n} = -\psi_{l,-n} = \psi_{-l,-n} = \tfrac{1}{4}\hat{\psi}_{l,n},\\ \theta_{l,n} &= \theta_{-l,n} = -\theta_{l,-n} = -\theta_{-l,-n} = \frac{1}{4\pi^3}\hat{\theta}_{l,n},\end{aligned}\right\} \tag{3.4}$$

where the range of z has been extended to $-1 \leqslant z \leqslant 1$. Furthermore, all the coefficients $\psi_{l,n}$ and $\theta_{l,n}$ are real.

A simplification of notation will be introduced by regarding the pair of integers l, n as the components of a vector γ. We may then replace the double subscripts l, n, by a single subscript γ. The summation over γ is then understood to stand for summation over the components l and n.

The problem now is to determine the coefficients $\psi_{l,n}$ and $\theta_{l,n}$ so as to make (3.3) satisfy the two equations (2.8) and (2.10). Substituting (3.3) into these equations and equating to zero the coefficient of the individual components $\exp(lkx+nz)\pi i$ we obtain the following two systems of spectral equations

$$\alpha_\gamma^4\psi_\gamma - lk\theta_\gamma = -\sigma^{-1}kB_\gamma, \tag{3.5a}$$

$$\alpha_\gamma^2\theta_\gamma - lk\lambda\psi_\gamma = -kH_\gamma, \tag{3.5b}$$

where $\alpha_\gamma^2 \equiv l^2k^2+n^2$, $\lambda \equiv R/\pi^4$, and B_γ and H_γ are the vorticity–advection spectrum and the heat–advection spectrum, which represent the contribution to the γ-component by the non-linear interactions between the various wave-number vector pairs γ_1 and γ_2 in the equations (2.8) and (2.10). These two spectral functions are given by

$$B_\gamma = \sum_{\gamma_1}\sum_{\gamma_2}(l_1n_2-l_2n_1)\,\alpha_{\gamma_2}^2\psi_{\gamma_1}\psi_{\gamma_2}, \tag{3.6a}$$

$$H_\gamma = \sum_{\gamma_1}\sum_{\gamma_2}(l_1n_2-l_2n_1)\,\psi_{\gamma_1}\theta_{\gamma_2}, \tag{3.6b}$$

where the wave number pairs γ_1 and γ_2 satisfy the selection rule

$$\gamma_1+\gamma_2 = \gamma, \quad \text{i.e.} \quad l_1+l_2 = l, \quad n_1+n_2 = n. \tag{3.7}$$

Because of this relation, we may replace γ_1 by $\gamma-\gamma_2$ and write the sum over γ_2 only, which is from $-\infty$ to $+\infty$ for both l and n. However, for each interacting pair γ_1, γ_2 there is a corresponding pair γ_2, γ_1. Both of these interacting pairs must be included in these advection spectral functions. Therefore we may write (3.6a and b) in the following non-redundant forms

$$B_\gamma = \sum_{\gamma_2} (ln_2-l_2n)\,(\alpha_{\gamma_2}^2-\alpha_{\gamma-\gamma_2}^2)\,\psi_{\gamma-\gamma_2}\psi_{\gamma_2}, \tag{3.8a}$$

$$H_\gamma = \sum_{\gamma_2} (ln_2-l_2n)\,(\psi_{\gamma-\gamma_2}\theta_{\gamma_2}-\psi_{\gamma_2}\theta_{\gamma-\gamma_2}), \tag{3.8b}$$

where the relation (3.7) has been used.

The two equations (3.5a and b) together with the relations (3.8a and b) form a closed system for the joint determination of ψ_γ and θ_γ.

It may be mentioned that the perturbation temperature θ defined by (2.9) includes a part $\bar{\theta}(z)$ which is independent of x. This part represents the modification of the mean temperature distribution by convection, and is given by the components with $l = 0$ in (3.3). On the other hand, no mean wind is produced by the convention, therefore $\psi_\gamma = 0$ when $l = 0$.

In solving the two sets of equations (3.5a and b) for $l \neq 0$, we may eliminate θ_γ and obtain

$$(\lambda-\lambda_\gamma)\,\psi_\gamma = (1/l)\,H_\gamma+(\alpha_\gamma^2/l^2k\sigma)\,B_\gamma \quad (l \neq 0), \quad \lambda_\gamma \equiv \alpha_\gamma^6/l^2k^2. \tag{3.9}$$

Denoting θ_γ by θ_{0n} and H_γ by H_{0n} when $l = 0$ and substituting θ_γ from (3.5a) we find

$$n\theta_{0n} = -(k/n)\,H_{0n} = \sum_{\gamma_1} l_1 k\psi_{n-\gamma_1}\theta_{\gamma_1}. \tag{3.10}$$

It appears easier to enumerate the the terms of H_γ by using (3.5a) and (3.10) as auxiliary equations, rather than to eliminate θ_γ completely.

4. The advection spectra $\mathbf{B}_\gamma$ and $\mathbf{H}_\gamma$

Since B_γ and H_γ are given by the sums of infinite numbers of non-linear interactions, the first step toward solving the infinite sets of equations (3.9) and (3.10) is to find the most important terms of these advection spectra.

As has been discussed before, the spectral equations (3.9) and (3.10) are to be solved by expanding ψ_γ and the other quantities into power series of a parameter η, of the form

$$\psi_{l,n}(\eta) = \psi_{l,n,r}\eta^r+\psi_{l,n,r+2}\eta^{r+2}+\ldots, \tag{4.1}$$

where $\psi_{l,n,r+2j}$ represents a numerical coefficient. We define the order of magnitude r of ψ_r and θ_γ as the lowest power of η in their expansions. Since convection starts in the form $\psi_{1,1}$, it will be taken as a first-order quantity so that $r = 1$ for $\psi_{1,1}$. On the other hand, all the functions of the higher modes must be of higher order by definition because they are produced by non-linear interactions represented by B_γ and H_γ.

In the calculations which are presented in this paper, the expansions are not carried out beyond the eighth order, the components l, n, and the power $p = r+2j$ used are all less than the integer 9. Consequently, only one digit is required to represent l, n, and p. Therefore we shall omit the comma between these indices. This simplification of notation also applies to the coefficients in (4.1). It will be understood in the following that whenever a three-digit subscript is attached to a quantity, the first two digits represent the components l and n of the wave-number vector γ while the third digit p indicates that it is the coefficient of η^p in the power-series expansion. For example, ψ_{111} and ψ_{113} stand for the coefficients of η and of η^3 of the expansion of $\psi_{11}(\eta)$, respectively.

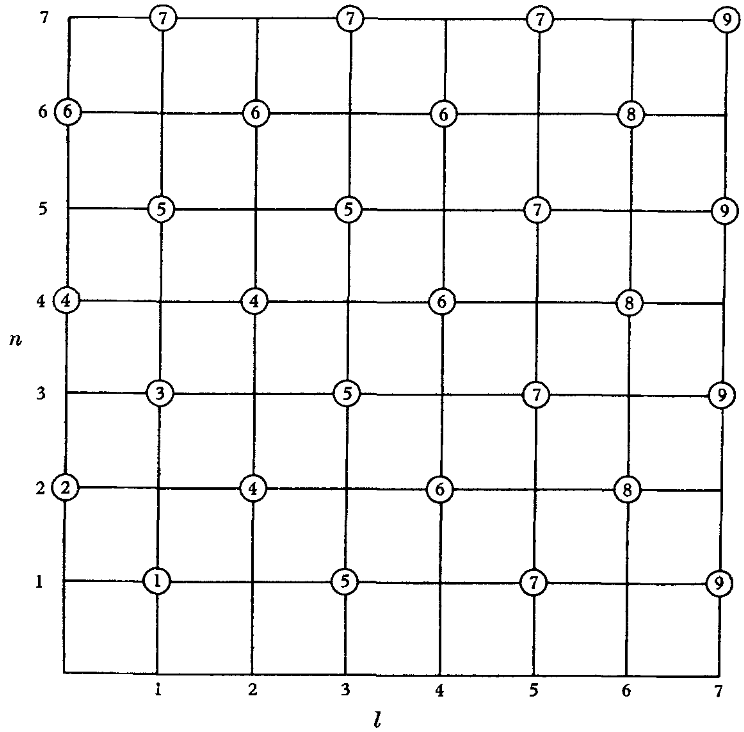

FIGURE 1. The order of magnitude of the spectral elements ψ_{ln}, θ_{ln} that occur in the non-linear solution as a function of the wave-numbers l, n.

The various components that are created by the non-linear interactions beginning with the initial convection ψ_{11}, θ_{11} are shown in figure 1, with their respective order of magnitude indicated by the numbers inside the circles. We note that for this case odd l occurs with odd n and even l occurs with even n, so that for every mode (l, n), $l+n$ is always even. We also note that for the modes with odd l and odd n, the power-series expansion is odd in η, while those modes with even l and even n are even functions of η. Therefore in the expansions represented by (4.1) the powers of η of two consecutive terms increase by 2.

As has been mentioned before, the first step toward solving the infinite sets of equations (3.9) and (3.10) is to find the most important terms of the advection spectral functions B_γ and H_γ. That is to say, our first task is to find the interacting pairs (γ_1, γ_2) that contribute to the lower-order terms of B_γ and H_γ.

The result of this portion of the work is given in Appendix I, where all the interacting pairs (γ_1, γ_2) that contribute to the γ-mode spectral functions B_γ and H_γ up to their η^7 or η^8 terms have been listed. In order to facilitate the reading, this table is given as an appendix at the end of this paper.

With the help of this table, it is a simple matter to express $\theta_{0,2n}$, B_γ and H_γ in terms of ψ_γ and θ_γ. Some of these non-linear spectral functions used in this study are given in Appendix II.

5. Expansion of spectral functions

In the paper by Kuo & Platzman referred to before, the spectral functions ψ_γ were expressed as infinite power series of a parameter $\Delta = (\lambda - \lambda_0)^{\frac{1}{2}}$. The function ψ_{11} so obtained is an alternating series whose rate of convergence is very slow and whose first four partial sums behave divergingly for $\lambda > 3\lambda_0$ (see figure 2*b*). However the oscillatory nature of this expansion suggests that an asymptotic solution which is valid for a larger range of values of λ can be obtained by choosing a more suitable parameter. In this paper we shall choose an expansion parameter η defined by

$$\eta = \left(\frac{\lambda - \lambda_0}{\lambda}\right)^{\frac{1}{2}}, \tag{5.1}$$

where λ_0 is the critical value of λ above which convection exists. We note that this expansion parameter η remains less than 1 for all finite values of λ. The advantage of using such an expansion parameter is to bring out the most important part of the solution in the lower-order terms and thereby to eliminate the oscillation of the solution.

The procedure of solving the spectral equation (3.9) is to expand ψ_γ, B_γ and H_γ in power series of η in the form shown in (4.1). In order to determine the various coefficients $\psi_{l,n,p}$ from the system of equations (3.9), we must also expand λ in power series of η. According to the definition (5.1) we have

$$\lambda = \frac{\lambda_0}{1-\eta^2} = \lambda_0\left(1 + \sum_{j=1}^{\infty} \eta^{2j}\right), \tag{5.2}$$

which shows that an infinite number of terms are required to represent λ. To overcome this difficulty we rewrite (5.2) in the form of a finite series by introducing a quantity λ_{0s}.† Thus

$$\lambda = \lambda_0 + \lambda_{0s}(\eta^2 + \eta^4 + \dots + \eta^{2s}), \tag{5.3}$$

$$\lambda_{0s} = \frac{\lambda_0}{1-\eta^{2s}}, \tag{5.3a}$$

where the integer s stands for the number of terms of the expansion. Since only powers of η lower than $2s$ occur in the formal expansion, the quantity λ_{0s} may be

† As will be seen from the subsequent development, we must take the first term of (5.3) as λ_0, not λ_{0s}. That is to say, we represent $(\lambda - \lambda_0)$ by a finite power series in η^2. Note that when s approaches infinity, λ_{0s} approaches λ_0.

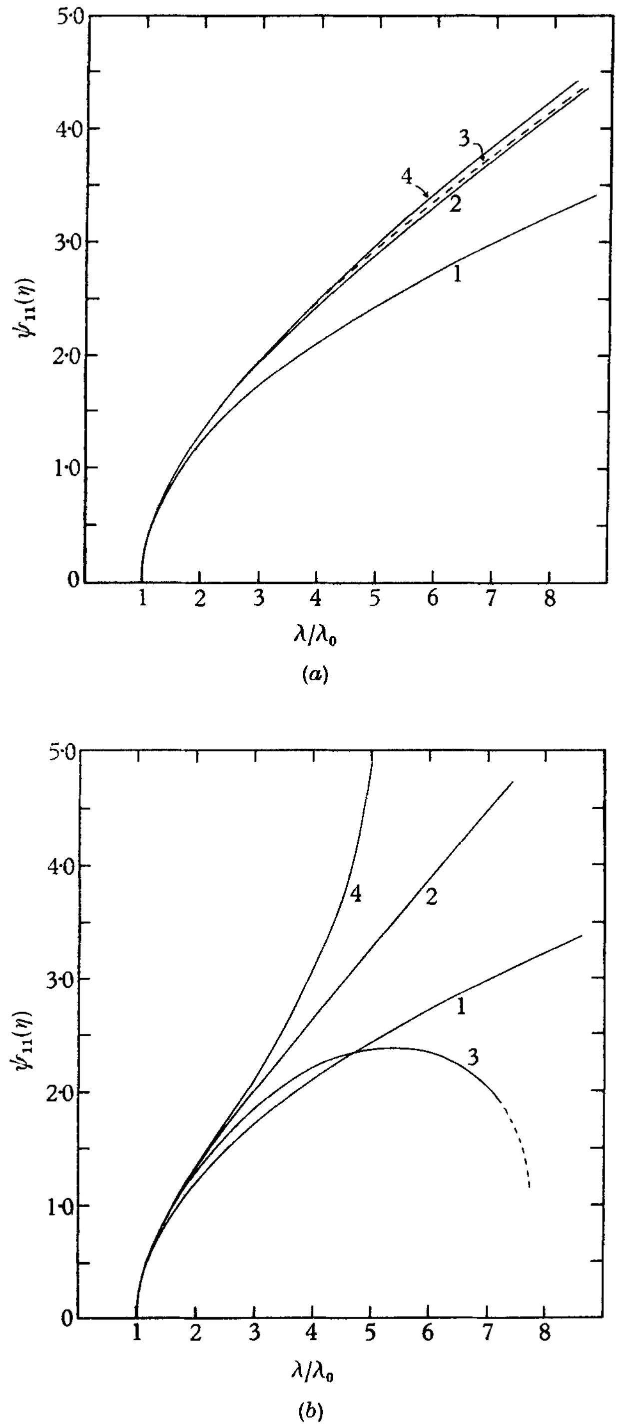

FIGURE 2. (*a*) Variation of the amplitude of the fundamental mode ψ_{11} as a function of λ/λ_0.
(*b*) Variation of ψ_{11} as a function of λ/λ_0 given by the η_1-expansion. [$\eta_1^2 \equiv \epsilon = (\lambda/\lambda_0) - 1$.]

treated as a constant without affecting the expansion. On the other hand, in computation λ_{0s} is to be evaluated directly from the definition (5.3*a*) for any chosen value of s so that the representation of λ by (5.3) is exact.

The convergence of the expansion is to be tested by increasing s and observing the behaviour of the results.

Substituting (4.1) and (5.3) into (3.9) and equating to zero the coefficients of the individual powers of η we obtain the following set of equations

$$(\lambda_\gamma - \lambda_0)\,\psi_{\gamma,\,r+2j} = -\frac{1}{l}H_{\gamma,\,r+2j} - \frac{\alpha_\gamma^2}{l^2 k\sigma}B_{\gamma,\,r+2j} + \lambda_{0s}\sum_{p=0}^{j-1}\psi_{\gamma,\,r+2p}, \tag{5.4}$$

where $j = 0,\ 1,\ 2, \ldots,\ \lambda_\gamma \equiv (l^2k^2+n^2)^3/l^2k^2$, and λ_{0s} is given by (5.3*a*), which depends upon the number of terms s of the expansion of the individual ψ_γ. The coefficient $\theta_{\gamma,\,r+2j}$ of the θ_γ-expansion is given by

$$lk\theta_{\gamma,r+2j} = \alpha_\gamma^4\,\psi_{\gamma,r+2j} + kB_{\gamma,r+2j}/\sigma. \tag{5.4a}$$

The expansion coefficients $H_{\gamma,r+2j}$ and $B_{\gamma,r+2j}$ are expressed in terms of the expansion coefficients ψ_γ and θ_γ through the equations (3.8*a* and *b*).

To solve the system of equations (5.4) we at first put $\gamma = 1, 1, r = 1$ and $j = 0$ in (5.4). Since $B_{111} = H_{111} = 0$ and $\psi_{111} \neq 0$, the first result obtained from this equation is

$$\lambda_0 = \lambda_{11} \equiv (k^2+1)^3/k^2, \tag{5.5}$$

which gives the critical value λ_0 as a function of k^2. The minimum value of λ_0 is 6·75, corresponding to $k^2 = \frac{1}{2}$.

It may be pointed out that even though the solution can be developed for arbitrary values of k with the exception of some discrete values, only that solution corresponding to $k^2 = \frac{1}{2}$, or its neighbourhood, can be stable. However, in order to examine whether some slight change of the horizontal scale of convection will occur at a later stage, the solution has been carried out to the third approximation for the arbitrary k.

Because of the result (5.5) the term on the left of equation (5.4) disappears for $\gamma = 1, 1$. Therefore for the first mode this equation degenerates into the following

$$\lambda_{0s}\sum_{j=1}^{p}\psi_{11,\,2j-1} = H_{11,\,2p+1} + \frac{\alpha_{11}^2}{k\sigma}B_{11,\,2p+1} \quad (p = 1, 2, 3, \ldots). \tag{5.6}$$

The expansion coefficients of the first mode must be obtained from this equation while those of the higher modes are given by (5.4).

We shall demonstrate the way of development of the solution by obtaining the first few expansion coefficients.

Putting $p = 1$ in (5.6) gives

$$\lambda_{0s}\psi_{111} = H_{113} + (\alpha_{11}^2/k\sigma)\,B_{113}. \tag{5.7}$$

From the equations (3.8*a* and *b*) and (3.10), or from equation (A.4) in Appendix II we find

$$B_{113} = 0, \quad H_{113} = -2\psi_{111}\theta_{022}, \quad \theta_{022} = -\alpha_{11}^4\psi_{111}^2.$$

Therefore (5.7) yields the results

$$\theta_{022} = -\tfrac{1}{2}\lambda_{0s}, \tag{5.8}$$

$$\psi_{111} = (\tfrac{1}{2}\lambda_{0s})^{\frac{1}{2}}/\alpha_{11}^2, \quad k\theta_{111} = \alpha_{11}^4\psi_{111}. \tag{5.9}$$

By taking the plus sign here, we have chosen $x = 0$ at the point where there is ascending motion.

With $\gamma = 1, 3$ and $r = 3, j = 0$, equation (5.4) reduces to

$$(\lambda_{13}-\lambda_{11})\,\psi_{133} = -H_{133}-(\alpha_{13}^2/k\sigma)\,B_{133}. \tag{5.10}$$

From (A. 5) and (5.8) we find

$$B_{133} = 0, \quad H_{133} = -\lambda_{0s}\psi_{111}.$$

Substituting in (5.10) and (5.4a) then yields the results

$$\psi_{133} = \frac{\lambda_{0s}}{\lambda_{13}-\lambda_{11}}\psi_{111}, \quad k\theta_{133} = \alpha_{13}^4\psi_{133}. \tag{5.11}$$

To obtain ψ_{113} we put $p = 2$ in (5.6). It then gives

$$\psi_{113}+\psi_{111} = \frac{1}{\lambda_{0s}}\left(H_{115}+\frac{\alpha_{11}^2}{k\sigma}B_{115}\right). \tag{5.12}$$

From equations (A. 1) and (A. 4) in Appendix II we find

$$\theta_{024} = -2\alpha_{11}^4\psi_{111}\psi_{113}+(\alpha_{11}^4+\alpha_{13}^4)\,\psi_{111}\psi_{133},$$

$$B_{115} = 0, \quad H_{115} = 3\lambda_{0s}\psi_{113}-\{2+(\alpha_{13}^4/\alpha_{11}^4)\}\,\lambda_{0s}\psi_{133}.$$

Substituting into (5.12) and making use of (5.11) and (5.4a) we obtain

$$\psi_{113} = \tfrac{1}{2}\psi_{111}\left\{1+\frac{\lambda_{0s}}{\lambda_{13}-\lambda_{11}}\left(2+\frac{\alpha_{13}^4}{\alpha_{11}^4}\right)\right\}, \tag{5.13}$$

$$\theta_{024} = -\tfrac{1}{2}\lambda_{0s}\left(1+\frac{\lambda_{0s}}{\lambda_{13}-\lambda_{11}}\right). \tag{5.14}$$

At this stage the coefficient θ_{044} can be determined also. From the equations (A.2), (5.9) and (5.11) we find

$$\theta_{044} = -\frac{\lambda_{0s}^2}{4(\lambda_{13}-\lambda_{11})}\left(1+\frac{\alpha_{13}^4}{\alpha_{11}^4}\right). \tag{5.15}$$

From the above developments we see that the coefficients $\psi_{11,r}$ and $\theta_{02,r+1}$ must be determined simultaneously.

The higher-order coefficients can be obtained in the same manner. However, the calculation becomes more tedious as the order increases, especially when k is being kept arbitrary. In reality, one k value must be selected by the fluid according to certain physical principles. The purpose of using an arbitrary k is that it enables us to investigate the probability of a change of scale of the convection at some later stage of the development, for example, by applying a certain selection principle to our non-linear solution which contains an arbitrary k. From the theoretical point of view, a comparison of the relative stability of the solutions will provide the proper selection rule. However, an analysis of the relative stability of the non-linear solutions is very difficult to carry out, and we are therefore forced to make use of more heuristic principles. One such principle is to maximize the total kinetic energy. This is based on the consideration that the final equilibrium state is arrived at through steps of development, each of which is represented by a maximum rate of increase of the kinetic energy, and therefore the end result should be characterized by a maximum of the total kinetic energy.

Another heuristic selection principle is the maximization of the heat transport, which has been used by Malkus & Veronis (1958).

As has been mentioned earlier, the cell scale which is selected on the basis of the linear theory (minimum λ_0) for two free boundaries is $k^2 = \frac{1}{2}$. This value of k, or some near-by value, must be taken as the starting scale of convection because other values of k represent unstable conditions. On the other hand, when λ is well above the lowest critical value, a change of the horizontal scale may occur. However, no such shift is found by either one of the two selection principles mentioned above for $\lambda \leqslant 1{\cdot}5\lambda_0$. For still higher values of λ the determination of the most preferred scale is very difficult because of the extreme flatness of the heat transfer and kinetic energy functions around $k^2 = \frac{1}{2}$. At any event, such slight shift of the horizontal scale is insignificant for the energetics of the system. We shall therefore restrict the higher-order expansions and the subsequent developments to $k^2 = \frac{1}{2}$ only.†

The expansion has been carried out to the η^7- and the η^8-order terms. Since the higher-order coefficients are polynomials of both q ($\equiv \lambda_{0s}/\lambda_0 = 1/1-\eta^{2s}$) and σ, their forms are very complex. We have therefore listed them in Appendix III. However, in order to illustrate the nature of our expansion, here we shall give the expansions of the two most important and also most interesting spectral functions ψ_{11} and θ_{02}:

$$\begin{aligned}\psi_{11} = {} & 1{\cdot}2247q^{\frac{1}{2}}(\eta + \tfrac{1}{2}\eta^3 + \tfrac{3}{8}\eta^5 + \tfrac{5}{16}\eta^7) \\ & + 0{\cdot}1019q^{\frac{3}{2}}(\eta^3 + \tfrac{3}{2}\eta^5 + \tfrac{15}{8}\eta^7) - 0{\cdot}0117q^{\frac{5}{2}}\eta^7 \\ & - 10^{-3}q^{\frac{5}{2}}(27{\cdot}5 + 3{\cdot}4\sigma^{-1} + 0{\cdot}85\sigma^{-2})(\eta^5 + 2{\cdot}088\eta^7) \\ & + 10^{-4}q^{\frac{7}{2}}(153 + 42\sigma^{-1} + 31\sigma^{-2} + 15\sigma^{-3} + 4\sigma^{-4})\eta^7 + \ldots, \end{aligned} \tag{5.16}$$

$$\begin{aligned}\theta_{02} = {} & -3{\cdot}375q(\eta^2 + \eta^4 + \eta^6 + \eta^8) - 0{\cdot}013338q^2(\eta^4 + 2\eta^6 + 3\eta^8) \\ & - 10^{-3}q^3\eta^6(10{\cdot}28 - 3{\cdot}97\sigma^{-1} - 2{\cdot}30\sigma^{-2}) \\ & - 10^{-3}\eta^8[(3{\cdot}085 - 4{\cdot}73\sigma^{-1} - 5{\cdot}00\sigma^{-2})q^3 \\ & - (1{\cdot}40 + 1{\cdot}86\sigma^{-1} + 1{\cdot}11\sigma^{-2} + 2{\cdot}18\sigma^{-3} + 1{\cdot}01\sigma^{-4})q^4] + \ldots. \end{aligned} \tag{5.17}$$

We mention again that if these equations are to be used up to the η^7 and the η^8 terms, we must then put $s = 4$ in q. On the other hand, if equation (5.16) is used only up to the η^5 terms, we should then put $s = 3$ in q.

We note that in the present problem, the Prandtl number σ appears only in the higher-order coefficients with $r \geqslant 4$, and that its direct effect on ψ is small when σ is not much smaller than unity. The effect of σ on the temperature field is even smaller.

6. Variations of the spectral functions with λ

The values of the spectral coefficients ψ_γ and θ_γ have been computed from the fourth approximations of the η-expansion for $\sigma = 10$ and for different values of λ, up to $\lambda = 8\lambda_0$. Within this range, the solutions for ψ_{11} and θ_{11} converge rapidly,

† Expansion coefficients for an arbitrary k have been obtained up to the η^5 and η^6 terms and are given in Scientific Report No. 3 of the M.I.T. Planetary Circulation Project, October 1960.

as can be seen from figure 2*a*, where the values of ψ_{11} computed from the first four approximations (corresponding to $s = 1, 2, 3, 4$) have been plotted. In fact, the values of ψ_{11} given by the third approximation are correct to the third significant figure for $\lambda \leqslant 3\lambda_0$, and the differences between the third and the fourth approximations remain less than 3 % in the whole range of computation.

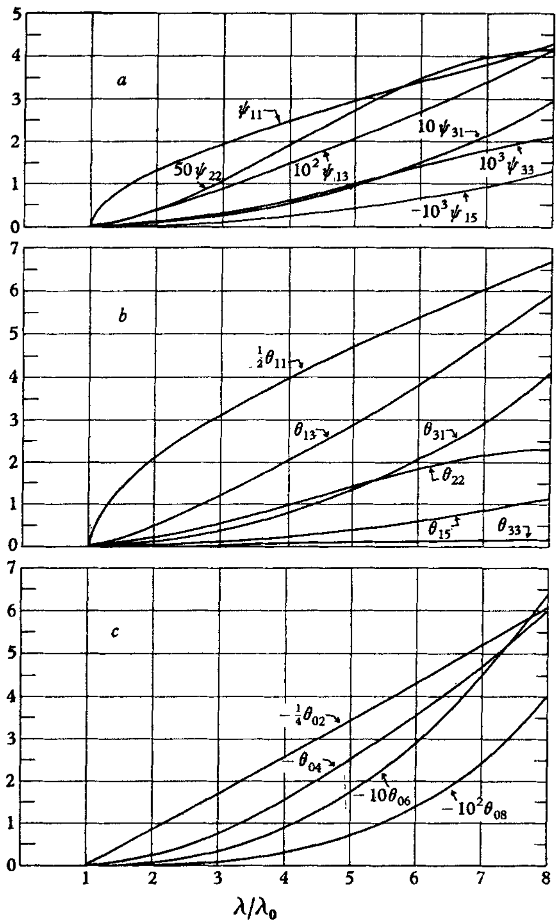

FIGURE 3. The variations of the spectral functions ψ_γ and θ_γ as functions of λ/λ_0.

For comparison, the values of ψ_{11} given by the first four approximations of the Δ-expansion ($\Delta^2 \equiv \lambda - \lambda_0$) obtained in the paper by Kuo & Platzman (1960) have been plotted in figure 2*b*. It is seen that the first four partial sums so obtained converge only for $\lambda < 2\lambda_0$, whereas for $\lambda > 3\lambda_0$ they behave divergently. A similar behaviour has been exhibited by the solutions of Malkus & Veronis (1958).

The second quantity that converges rapidly in the present solution is θ_{02}. The differences between the first four approximations remain less than 3 % within the whole range of computation. Its values also agree well with the values obtained directly from (A. 1) by making use of the values of ψ_γ and θ_γ, which shows that the solution is self-consistent to a relatively high degree of accuracy. On the other hand, the rates of convergence for the coefficients of the higher modes are much slower for larger λ.

The values of the various spectral functions as given by their respective fourth approximations have been plotted in figure 3 for different values of λ/λ_0, where some proper constant factors have been introduced so as to make them have the same order of magnitude.

From the results in the figures 3 *a* and *b* we see that just a little above the critical point the various components are arranged according to their order of expansion. The intensity of the first mode increases very rapidly near $\lambda = \lambda_0$ whereas those of the higher modes increase only very slowly. However, this is not true for larger values of λ.

One significant result of the computation is that for larger values of λ, the magnitudes of the higher-order coefficients ψ_γ and θ_γ are not arranged according to their degrees in η. The curves in figure 3 show that some of these coefficients increase more rapidly than the others. For example, even though ψ_{31} is a fifth-order quantity, it becomes the largest higher mode of the ψ-field for $\lambda > 3\lambda_0$. The next largest is ψ_{22} while ψ_{13} ranks third. This is also true in the kinetic energy spectrum, which is given by $\alpha_\gamma^2 \psi_\gamma^2$.

In the temperature spectrum, θ_{31} also increases more rapidly than the other components even though θ_{13} remains the largest higher mode in the range of computation. As a result of the rapid growth of the (3, 1)-mode of convection, it becomes the second largest contributor to the heat transfer for $\lambda > 2\lambda_0$. We also mention that at $\lambda = 8\lambda_0$, θ_{31} and θ_{13} are of the same order of magnitude as θ_{11}. The rapid rate of growth of ψ_{31} and θ_{31} for large values of λ is due to their large expansion coefficients in equation (5.4), which are proportional to $(\lambda_\gamma - \lambda_{11})$. Judging from equation (5.4) which holds for all the higher modes γ, we may expect that at larger values of λ, the magnitudes of ψ_γ will be roughly proportional to $(\lambda_\gamma - \lambda_{11})^{-1}$ while the contribution from the (l, n)-mode to the heat transfer is likely to become proportional to $\alpha_{ln}^4(\lambda_{ln} - \lambda_{11})^{-2}$.

7. Convective heat transport and mean temperature distribution

In the results obtainable from the non-linear solution, the dependence of the rate of heat transfer upon the imposed temperature difference is of primary interest. This dependence can most conveniently be expressed by the functional relation between the Nusselt number N and the Rayleigh number, or between the heat transfer ratio S and the Rayleigh number.

The Nusselt number is the ratio of the actual heat transport rate to the rate at which heat would be transported by conduction for the given temperature difference between the hot and cold reservoirs were convection absent. Thus, according to this definition N is given by

$$N = -\frac{h}{\Delta T}\left(\frac{d\overline{T}}{dz}\right)_{z=0} = -\frac{1}{R}\left(\frac{d\overline{T}}{dz}\right)_{z=0},$$

where $\overline{T}$ is the horizontally averaged temperature.

On the other hand, the heat transport ratio S is defined as the ratio between the actual heat transport to the rate at which heat would be transported conductively at the critical Rayleigh number, therefore it is given by

$$S \equiv \frac{R}{R_0} N = -\frac{1}{R_0} \left(\frac{d\overline{T}}{dz} \right)_{z=0}.$$

According to the equations (2.9) and (3.2), $\overline{T}$ is given by

$$\overline{T} = T_0 - Rz + \sum_{n=1}^{\infty} \hat{\theta}_{0,2n} \sin(2\pi nz).$$

Substituting $\theta_{0,2n}$ from (3.4) we obtain

$$\begin{aligned} \overline{\Theta} &\equiv (\overline{T} - T_0)/R \\ &= -z + \frac{2}{\pi\lambda} \sum_{n=1}^{\infty} \theta_{0,2n} \sin(2\pi nz). \end{aligned} \tag{7.1}$$

Therefore the heat transport ratio S is given by

$$S = \frac{\lambda}{\lambda_0} - \frac{4}{\lambda_0} \sum_{n=1}^{\infty} n\theta_{0,2n}. \tag{7.2a}$$

The values of the first four $\theta_{0,2n}$ have been obtained from their η^8-approximations and are plotted in figure 3*c*. According to equation (3.10), $\theta_{0,2n}$ is produced by the non-linear interactions between the various modes. However, it can be seen from the equations following (A. 1) that only the non-linear interaction between ψ_γ and θ_γ of the individual modes themselves contributes to the heat transport, while the non-linear interactions between different modes cancel out in equation (7.2*a*). Thus, in terms of ψ_γ and θ_γ, S is given by

$$S = \frac{\lambda}{\lambda_0} + \frac{4k}{\lambda_0} \sum_{\gamma} l\psi_\gamma \theta_\gamma. \tag{7.2b}$$

Thus there are two different ways of computing S (or N). One method is by taking the sum of the various $n\theta_{0,2n}$ as given by their respective expansions (figure 3). The other method is to compute S directly from the second expression (7.2*b*) with the values of ψ_γ and θ_γ obtained from the solution. This method represents a higher approximation than the first, but not completely in accordance with the order of the η-expansion.

The variations of S with λ as given by the second-order and the eighth-order expansions are represented in figure 4 by the curves $S^{(1)}$ and $S^{(4)}$, respectively, while $S^{(d)}$ is obtained from the second expression (7.2*b*) and the ψ_γ and θ_γ functions given in figure 3. It is seen that for larger λ/λ_0, $S^{(4)}$ is much higher than $S^{(1)}$. On the other hand, the fourth-order and the sixth-order approximations of S are nearly the same as $S^{(4)}$, therefore they are not plotted in these graphs.

It may be mentioned that the Δ-expansion ($\Delta^2 = \lambda - \lambda_0$) of the Nusselt number obtained by Kuo & Platzman (1960) also diverges for $\lambda > 3\lambda_0$.

On examining the various terms of $S^{(d)}$ it is found that the (3.1) mode makes an appreciable contribution to the heat transport when λ is greater than $4\lambda_0$. Since this transport is of the tenth order in η, its effect is not included in the eighth-order approximation $S^{(4)}$. Thus, the difference between $S^{(d)}$ and $S^{(4)}$ is mainly due to the transport by the (3, 1)-mode.

It may be mentioned that the second-order approximation to S in this η-expansion is identical with the corresponding second-order approximation of the Δ-expansion. This approximation gives a linear dependence of the convective heat transport on the Rayleigh number. On the other hand, the results given by the higher approximations obtained in this study definitely show that the heat transport increases faster than the first power of λ.

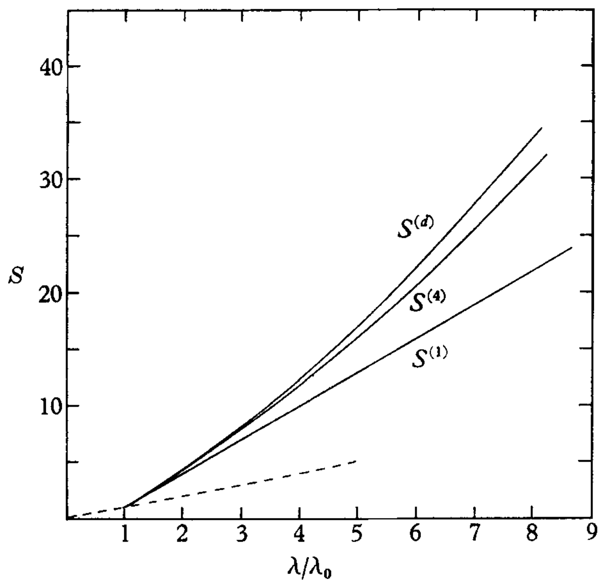

FIGURE 4. The variation of the heat transport function S as a function of λ/λ_0.

On plotting the quantity $\log(S-1)$ against $\log\{(\lambda/\lambda_0)-1\}$, we find that for $\lambda > 4\lambda_0$, the $S^{(d)}$ and $S^{(4)}$ obtained above may also be represented by the formula

$$S = 1+c\{(\lambda/\lambda_0)-1\}^{\alpha}. \tag{7.3}$$

The value of the exponent α as given by $S^{(d)}$ is $\alpha = 1{\cdot}24$ while that given by $S^{(4)}$ is $\alpha = 1{\cdot}19$. These values are very close to the observational result $S \sim (\lambda/\lambda_0)^{\frac{5}{4}}$ for laminar convection (see Jakob 1949).

We mention that the convection heat transfer given by Nakagawa's heuristic theory (1960) is the same as that given by the second-order approximation obtained in the present paper. Evidently that approximation greatly underestimates the convective heat transfer except when λ is only slightly above its critical value.

Besides the heat transport, the next most interesting quantity obtainable from the non-linear solution is the modified mean temperature distribution, which is given by (7.1). The mean temperature profile has been computed from this equation for six different values of λ, by making use of the $\theta_{0,2n}$ values given in figure 3. The results of those computations are represented in figure 5 by the six curves, where the numbers attached to these curves indicate the values of λ/λ_0.

These curves show clearly the effect of convective heat transfer on the mean temperature distribution. The most striking feature of this effect, as revealed by the solution, is that for $\lambda > 2\lambda_0$ a region of isothermal stratification is produced by convection in the middle of the fluid layer.† The thickness of the isothermal layer increases as λ increases, so that when the imposed temperature difference

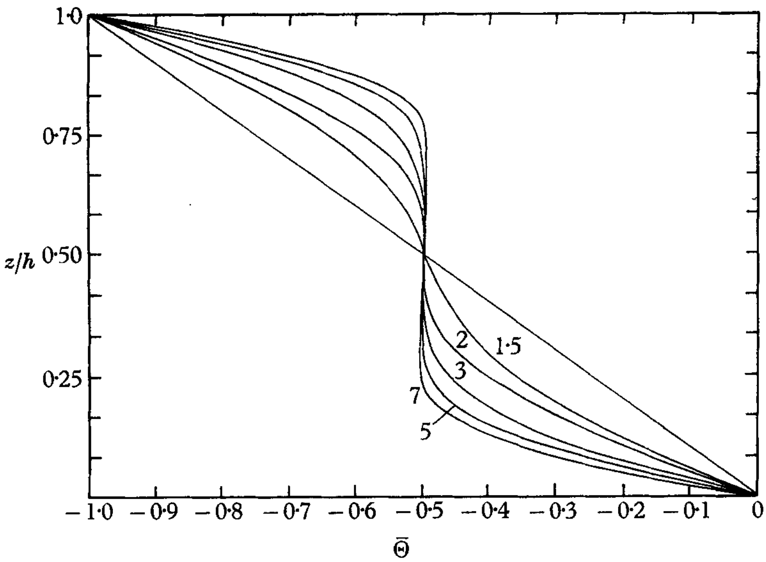

FIGURE 5. Distribution of the mean temperature for different values of λ/λ_0.

between the hot and cold sources is very large, the main body of the fluid will become isothermal (or adiabatic if the medium is a gas) in the mean, while large mean temperature gradients will be confined to the boundary layers adjacent to the hot and cold reservoirs thus providing a higher rate of heat transport. This feature of the development must persist when the convection becomes turbulent, even though the solution obtained here does not hold in the turbulent régime. This paradoxical result of the creation of a deep isothermal (adiabatic) layer by convection at a large imposed temperature difference is very important for the understanding of the convective processes in nature, where observations usually give a critical temperature gradient instead of a supercritical gradient.

It may be pointed out that even though the non-linear interactions between different modes of convection do not contribute to the total heat transport, they are very important for the modification of the mean temperature distribution, because these terms do not cancel except at the boundaries.

8. Pattern of isotherms

Since the temperature θ is a cosine series in x and therefore is symmetric with respect to $x = 0$, it is sufficient to consider only a half cell. We therefore

† The coefficients $\theta_{0,2n}$ in figure 3c give a very small reversed gradient near $z = \frac{1}{2}$ for $\lambda \geqslant 3\lambda_0$, which is numerically much less than the errors of the approximations and therefore should not be taken as a real feature.

restrict attention to the region $0 \leqslant \xi \leqslant \pi$ and $0 \leqslant \zeta \leqslant \pi$, where $\xi = k\pi x$ and $\zeta = \pi z$. Expressed in terms of $\Theta \equiv (T - T_0)/R$, we have

$$\Theta = \overline{\Theta} - \frac{4}{\pi\lambda} \sum_{\gamma} \theta_\gamma \cos l\xi \sin n\zeta,$$

where $\overline{\Theta}$ is the horizontal average of Θ and is given by equation (7.1). It can easily be shown that Θ is diagonally antisymmetric about the point $\xi = \frac{1}{2}\pi$, $\zeta = \frac{1}{2}\pi$.

In order to demonstrate the effects of the higher-order terms we take as our example the solution for $\lambda = 7\lambda_0$. For this λ we find that the values of the coefficients θ_{ln} for $\sigma = 10$ are

$$\begin{aligned}
&\theta_{11} = 12{\cdot}145, &&\theta_{13} = 4{\cdot}810, &&\theta_{15} = 0{\cdot}831, &&\theta_{17} = 0{\cdot}066;\\
&\theta_{22} = 2{\cdot}160, &&\theta_{24} = -0{\cdot}298, &&\theta_{31} = 2{\cdot}900, &&\theta_{33} = -0{\cdot}060;\\
&\theta_{02} = -20{\cdot}72, &&\theta_{04} = -4{\cdot}670, &&\theta_{06} = -0{\cdot}448, &&\theta_{08} = -0{\cdot}024.
\end{aligned}$$

The other coefficients are very small, and therefore will be neglected. The isotherms given by this solution are shown in figure 6*a*. The most prominent features are the concentration of large temperature gradients in the boundary regions where the ascending and descending currents are approaching the boundaries, whereas in the boundary regions where these currents are moving away as well as in the main bodies of these currents the temperature gradients are very small. Another feature of the temperature distribution is the mushroom-like spreading of the ascending and descending currents, which is a characteristic feature of convection due to isolated hot bubbles, but also appears in this solution. The curve in figure 6*b* represents the temperature distribution at the level $z = \frac{1}{2}$, as given by the solution. It illustrates more clearly the effect of the higher mode (3, 1). This distribution is surprisingly similar to that given by Silveston (1958) which is deduced from observations.

Because the ψ_{ln} of the higher modes are much smaller than the ψ_{11}, the distortion of the streamline field is not so pronounced as that of the temperature field.

9. Discussion

There is no doubt that the non-linear solution obtained in the present work converges more rapidly and is valid over a much larger range of the imposed temperature difference than the solutions obtained in previous studies. It has also revealed many interesting features of the convective motion, such as the concentration of the mean temperature gradient in the boundary layers and the creation of a deep mean isothermal layer in the main body of the fluid, and the mushroom-like spreading of the warm-ascending and cold-descending currents.

A number of questions may be raised concerning this solution, such as its uniqueness and its most likely modification in order to embrace the turbulent régime.

Since this solution is based on the assumption that convection starts at the critical Rayleigh number R_0 in the form of the fundamental mode and proceeds to create higher modes through the non-linear cascade effect and to reach the equilibrium steady state, the solution can be unique only under these stated conditions, and only when this solution converges and is stable. The last qualification is necessary because if the solution represents an unstable state, then the motion must be replaced by another motion which is stable for the given value of λ. The solution which represents this new state of convection may be composed

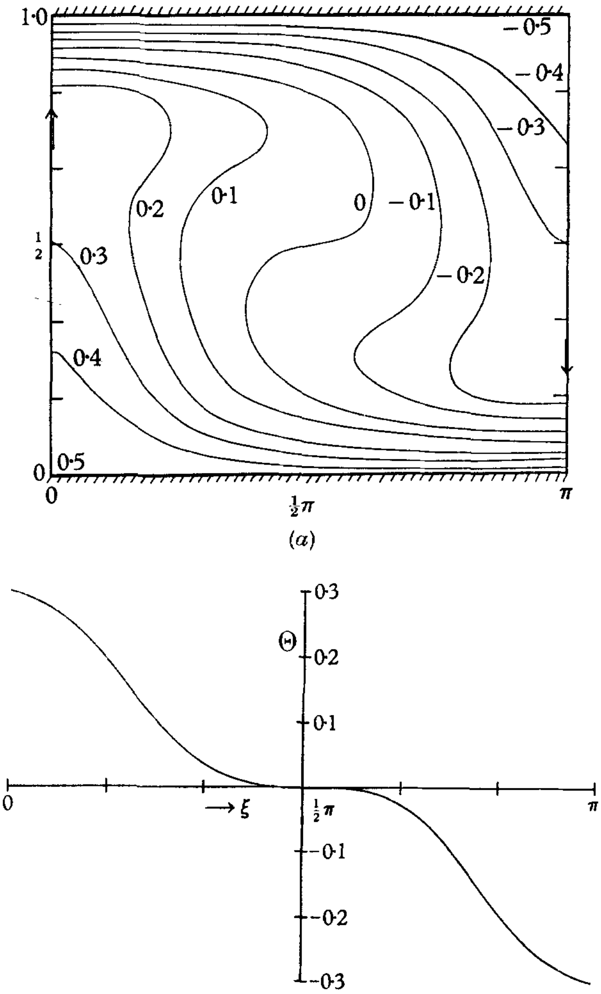

FIGURE 6. (*a*) The distribution of the total temperature as given by the non-linear solution for $\lambda = 7\lambda_0$ and $\sigma = 10$.

(*b*) The temperature distribution at the level $z = \frac{1}{2}$ as given by the solution.

of the same modes as given in figure 1, but with intensities different from that given by our present solution, or it could be composed of new components in addition to the modes in figure 1. In the former case the added parts of the various modes must appear in unsteady form, while in the latter case the new modes may either be steady or unsteady. In order to determine whether any new perturbation is needed, it seems necessary to examine the stability of the convective state represented by our non-linear solution.

Another point worth mentioning is that even though this solution is valid for a much larger range of temperature difference than the solutions obtained in the previous works, it also shows a tendency to divergence for values of λ greater than ten times the critical value. The reason for this divergence is apparently due to the fact that for large values of λ, the amplitudes of the higher modes ψ_{ln} increase as the factor $(\lambda_{ln}-\lambda_0)^{-1}$ and are no longer arranged according to their orders of appearances in the non-linear cascade effect depicted in figure 1. We note that for $\lambda > \lambda_{ln}$, we have

$$\frac{1}{\lambda_{ln}-\lambda_0} \simeq \frac{1}{\lambda-\lambda_0}\left\{1+\frac{\lambda-\lambda_{ln}}{\lambda-\lambda_0}\right\}.$$

Therefore for large λ the various modes behave somewhat like self-excited ones. It seems therefore that a different type of solution is needed for higher temperature differences, which may end up in the turbulent régime in which all modes are present.

A different solution of the non-linear equations which includes perturbations with horizontal scales longer as well as shorter than the fundamental mode can be obtained if the horizontal dimension L of the fluid is finite and is not an integral multiple of the horizontal scale a corresponding to the minimum critical Rayleigh number R_0. Then the motion must be delayed until λ reaches a value $\lambda_1(a_1)$ corresponding to the critical value for the horizontal scale a_1 such that $L = ma_1$, where m is an integer. We assume that λ_1 is the lowest λ which sustains convection which is compatible with the dimension L. Since a is the horizontal scale for which the critical λ_0 is an absolute minimum, there is another scale a_2 on the opposite side of a for which the critical value $\lambda_2(a_2)$ is equal to λ_1. Suppose $a_2 > a > a_1$ and $L = (m-1)\,a_2$. Then convection will start when λ reaches $\lambda_1 = \lambda_2$ and the initial motions must be represented by a fundamental mode consisting of two waves with horizontal wavelengths a_1 and a_2. In this case the non-linear interactions will produce new modes with both longer and shorter horizontal scales, and the spectrum will then be a complete spectrum. It seems that this consideration is important for the harmonic representation of the solution of the turbulent régime

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